

## Dispersive Methods and QCD Sum Rules for $\gamma\gamma$ Collisions

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### ABSTRACT

It has been shown, in the case of meson photoproduction, that the power-law falloff of these reactions can be described by lowest order (real) sum rules, at moderate momentum transfer. The phases of these processes, in this regime, are usually thought to be non-perturbative. In a sum rule framework, however, they can possibly be described by radiative corrections to the hadronic spectral densities of the corresponding helicities, which become complex functions to order  $\alpha_s$ , and the effects of interference can be strongly enhanced by the presence of the vacuum condensates in the dispersion relation. It is shown that the imaginary parts of these complex corrections have a factorized form and can be evaluated in a systematic fashion, while their real parts, at the same perturbative order, are down by at least 2 powers of momentum transfer. The analysis is done at two loop level, combining dimensional regularization and light-cone methods. The calculations are performed for all the independent set of scalar diagrams generated by the OPE. The analytical bounds are identified and discussed.

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# 1 Introduction

Sum rules have been a valuable tool in the investigation of the intermediate energy region of QCD. More recently, it has been suggested that such methods can find application in the context of elastic scatterings of Compton type [2]. Compared to sum rules for form factors, which involve dispersion relations only at fixed  $t$ , the description of elastic scattering has been performed by sum rule based on dispersion relations at both  $s$  and  $t$  fixed, and with a limited domain of analyticity in the remaining variables. For scatterings at intermediate angles such new description of 2-photon elastic processes compares favourably well with standard factorization theorems [4]. Such *extended* sum rule methods can be helpful in the attempt to understand on a firmer ground the transition to perturbative QCD in simple elastic processes, and can have potential application to a new entire class of direct reactions of Compton type and to the corresponding crossed channels.

In fact, two-photon collisions are among the most interesting ways to study these important aspects of elastic scattering at moderate momentum and energy transfers.

Together with Li, we have analyzed in detail the main features of the sum rules for pion Compton scattering and pion photoproduction [16] and compared their predictions with those derived from modified [7] [6] and standard factorization theorems [14].

All the work presented so far [3] [4] for these reactions is based on *real* sum rules for the two helicities  $H_1$  and  $H_2$ .

The analysis of the radiative corrections to the lowest order result are, however, of crucial importance in the study of the phases of these amplitudes, since nontrivial interference effects between those can be generated by such corrections. Differently from the usual prescriptions based on factorization theorems, such effects, in the sum rule context, can be strongly enhanced due to the presence of the vacuum condensates as fundamental parameters in the dispersion relations.

It is widely believed that in the intermediate energy region of QCD ( $Q^2 \approx 4-7 \text{ GeV}^2$ ) such phases cannot be incorporated in a direct perturbative treatment and are of non-perturbative origin. In the sum rule approach interference effects are generated by complex contributions (due to gluonic exchange) to the lowest order (real) spectral densities.

The evaluation of such corrections, however, is a nontrivial task and requires considerable effort. In this work (section 3) we discuss in great detail the properties of analyticity of the lowest order spectral densities of those correlators which interpolate with processes of Compton type [2]. The motivation of our study relies mainly on the fact that the region of analyticity of 4-point correlators, compared to that for the 3-point correlators which are commonly used in the sum rules for vertex functions, has unobvious features. Such a region turns out to be bound in size by  $u$ -channel singularities, a moving  $u$ -channel cut, whose location varies when the other variables which appear in the dispersion integral are varied. The study of the properties of analyticity of the spectral densities which appear in sum rules of these type is closely related to the mass parameters in the correlator of the 4-currents from which the OPE is calculated and the sum rules are derived.

A preliminary discussion of the mass dependence has been presented in ref. [3], where power corrections to the sum rule for a specific combination of the helicities of pion Compton scattering have been presented. An artificial mass dependence is generated in the calculation of such corrections (this point is briefly illustrated in section 4) and we have argued, in the same work, that this feature was not going to affect the size of the analyticity region or the validity of the sum rule, for small values of the mass.

Here we are going to give a general proof of these statements together with an explicit description of the analytical bound for the dispersion relation, generated by a finite mass dependence in the expansion of the correlator.

Such thresholds, absent in the case of form factors, cannot be ignored in the case of  $s$  and  $t$  dependent dispersion relations. This bound is given by a quadratic equation, and shows up as a singularity of the lowest order spectral density (the vanishing of its denominator). We show in sections 3 and 4 that moderate values of  $s$  and  $t$  keep the bound far away from the (finite) region in which the OPE is calculated. A similar (mass-dependent) bound appears for the diagrams describing the power corrections. The presence of these bounds is justified by the fact that each coefficient of the OPE has leading singularities (also called *Landau surfaces*) generated when all the internal lines in the perturbative expansion go on shell. For Compton scattering these surfaces corresponds to a  $u$  channel threshold. We show, however, that these singularities, though important in a more general analysis (in mass dependent correlators), disappear in the massless limit. Therefore, in this particular case, a simplified structure of the OPE emerges, which is fully exploited in the calculation of the radiative corrections. We are also able to write down a dispersive representation of the main integrals (the coefficients of the OPE) which is unbound in the plane of the two dispersive variables. The proof is limited to the lowest order result.

In this simplified case we prove that a limited applicability of Borel methods [1, 5] (Appendix C) is possible, and that the leading spectral density can be re-obtained elegantly without using the usual Cutkosky rules. This is consistent with our results of section 3.

The absence of these  $u$ -channel cuts (in the massless case) in the lowest order correlator is crucial for the evaluation of the radiative corrections. In fact we are looking for imaginary parts to the spectral densities and, to next order in perturbation theory, these cuts reappear in specific subdiagrams. The analysis presented in sections 3 and 4 is therefore preliminary to our subsequent discussion. In section 5 we illustrate, using various results of dispersion theory, how to organize the calculation of the radiative corrections to the spectral densities in a systematic way. The analysis of the complex parts in the diagrammatic expansion of the spectral density, is illustrated for all the independent set of diagrams, and a simple *factorized* structure (in terms of specific *sub-cuts*) of the OPE emerges.

Results of dispersion theory to lower order are then used to decide of the real or complex behaviour of such functions. Remarkably, thanks to this simple factorized feature, we show that such corrections can be evaluated in closed form. For this purpose, we combine dimensional regularization and light-cone methods, and show that the divergences in the OPE can be isolated as poles in  $\epsilon = n - 4$ .

Our conclusions are in section 6. Appendices A-C cover technical derivations of sections 3-5, while Appendix D illustrates, in a self contained way, how to extend Borel methods [1] to Compton scattering.

## 2 Spectral densities for 4-point correlators

Sum Rules relate the timelike region of a correlator to its spacelike part by a dispersion relation [1, 5, 12]. In the case of pion Compton scattering, for instance, the 4-point correlator with non vanishing projection over the invariant amplitudes (the helicities) of the process is given by [2]

$$\begin{aligned} \Gamma_{\sigma\mu\nu\lambda}(p_1^2, p_2^2, s, t) &= i \int d^4x d^4y d^4z \exp(-ip_1 \cdot x + ip_2 \cdot y - iq_1 \cdot z) \\ &\times \langle 0 | T \left( \eta_\sigma(y) J_\mu(z) J_\nu(0) \eta_\lambda^\dagger(x) \right) | 0 \rangle, \end{aligned} \quad (1)$$

where

$$J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d, \quad \eta_\sigma = \bar{u} \gamma_5 \gamma_\sigma d \quad (2)$$

are the electromagnetic and axial currents, respectively, of up and down quarks.  $q_1$  and  $q_2$  are on shell moment carried by the two physically polarized photons (see Fig. 1).

The two pion momenta are denoted by  $p_1$  and  $p_2$ , with  $s_1 = p_1^2$  and  $s_2 = p_2^2$  being the virtualities.

The invariant amplitudes of pion Compton scattering are then extracted from the physical expansion of this correlator for timelike momenta  $p_1, p_2$ . The matrix element which interpolates with pion states is given by [2]

$$M_{\nu\lambda} = i \int d^4y e^{-iq_1 y} \langle p_2 | T (J_\nu(y) J_\lambda(0)) | p_1 \rangle, \quad (3)$$

and can be related to the two helicities  $H_1, H_2$  as

$$M^{\lambda\mu} = H_1(s, t) e^{(1)\lambda} e^{(1)\mu} + H_2(s, t) e^{(2)\lambda} e^{(2)\mu}, \quad (4)$$

where  $e^{(1)}$  and  $e^{(2)}$  are helicity vectors defined in [2, 3].

The spectral densities appearing in the sum rules are characterized by a resonance contribution at the pion pole  $\Delta^{res}(s_1, s_2, s, t)$ , and by a continuum one  $\Delta^{pert}(s_1, s_2, s, t)$ , at large values of the virtualities  $s_1$  and  $s_2$  [2, 3]. A dispersion relation than allows us to connect the spacelike region of the 4-current correlator in eq. (2) to its timelike behaviour in  $p_1^2, p_2^2$  at fixed angle  $-t/s$ .

The resulting sum rule for  $H_i(s, t)$  ( $i = 1, 2$ ) can be written down in terms of a local duality contribution and of power corrections of quarks and gluons as

$$f_\pi^2 H_i(s, t) \left( \frac{s(s+t)}{-t} \right) = \left( \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho_i^{\text{pert}} + \frac{\alpha_s}{\pi} \langle G^2 \rangle \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho_i^{\text{gluon}} \right) e^{-(s_1+s_2)/M^2} + C_i^{\text{quark}} \pi \alpha_s \langle (\bar{\psi}\psi)^2 \rangle . \quad (5)$$

where  $s_0$ , the local duality interval, characterizes the resonant region of the correlator. The spectral densities  $\rho^{\text{pert}}$ ,  $\rho^{\text{quark}}$ ,  $\rho^{\text{gluon}}$  are calculated by the OPE. They are polynomial in a variable  $Q(s_1, s_2, t)$  which has a natural interpretation in the light-cone frame of the pion' lines (see Appendix A).

The evaluation of all the lowest dimensional contributions to the OPE of the two helicities has been presented in ref. [16].

For instance, the covariant expression of  $\rho_i^{\text{pert}}$  is of the form

$$\rho_i^{\text{pert}} = \frac{5}{24\pi^2} \frac{R_i(Q^2, s, s_1, s_2)}{Q^4(2Q^2 - s)^2(4Q^4 - s_1 s_2)^5(2Q^2 s - s_1 s_2)^2} , \quad (6)$$

where

$$R_i(Q^2, s, s_1, s_2) = \sum_{n=0}^{15} a_{i,n}(s, s_1, s_2) Q^{2n} . \quad (7)$$

The explicit forms of the coefficients  $a_{i,n}$  can be found in [16]. The contributions from the lowest dimensional vacuum condensates are also given in that work.

As shown below in eq. (39), the expression of  $Q(s_1, s_2, t)$  is non trivial and although it can be viewed as the basic momentum scale which characterizes the power law fall off of the spectral densities (see [3]), (which are simple rational functions when expressed in this variable), it is however not exactly on the same ground as  $s$  and  $t$  in characterizing the "dimensional counting" of the sum rules (as it is usually done in standard factorization [14]). There are two main reasons for this: 1)  $Q$  contains the sum of *both*  $s$  and  $|t|$  at larger energies (which in our case are supposed to be of the order of 5-7  $\text{GeV}^2$  or so [4]) 2) it appears as a variable (under the local duality interval) and therefore it is not just a fixed external invariant. The power law falloff of the spectral density, however, shows that [3, 16], generically

$$\rho(s_1, s_2, s, t) \approx \frac{C_1}{Q^6} + \frac{C_2}{Q^8} + \dots \quad (8)$$

where  $C_1$ ,  $C_2$ , etc, are complicated functions of the pion virtualities and contain the input from the condensates. Therefore, contributions from "higher-twists" are embodied in an obvious form by the dispersion relation. There are some specific features of this approach which clearly differentiate it from the perturbative picture: 1) to lowest order, no gluons are involved, 2) the

power law falloff is much more suppressed (a coherence effect) compared to the perturbative one and 3) the sum rule prediction starts dominating compared to the perturbative one toward the region of forward scattering. Features 1) and 2) are known from the literature on sum rules for form factors. However, in this latter case, the contributions from the condensates grows as a power of "Q" and soon overtakes the (decaying) contribution from the perturbative coefficient of the OPE. In our case, instead, a suppression is generated -for scattering at fixed angle- at the photon vertex by a virtual quark line connecting the two photons and eliminates this growth. In fact, compared to form factors, a dependence on both  $s$  and  $t$  is now available in the sum rule. The shortcome of this, however, is that the stability analysis of the sum rules (in both  $s$  and  $t$ ) becomes much more complex [16].

In order to describe, for instance, interference effects in photoproduction processes ( $\gamma \gamma \rightarrow \pi^+ \pi^-$ ) and in other related processes by a sum rule to order  $\alpha_s$ , one needs the imaginary parts of the radiative corrections to the spectral densities. The real parts are also, as a matter of fact, needed, however an actual calculation shows that these are suppressed by a factor  $1/Q^4$  compared to the imaginary ones. The argument is quite simple and can be better understood from a reading of section 5. Basically, imaginary parts are generated by pinched diagrams containing 5-particle cuts, while real parts require only 4-particle cuts. This implies that more propagators of large virtuality ( $1/Q^2$ ) appear in the integrals describing real discontinuities, compared to those contributing to the imaginary ones. One can also define suitable observable were the contributions from the imaginary parts are enhanced compared to the real ones.

Here is a simple example of how this may be happen.

If we include radiative corrections,  $H_1$  and  $H_2$  are determined by the *complex* spectral densities

$$\rho_i(s_1, s_2, s, t) = \rho_i^{(0)} + \alpha_s \left( \rho_i^{(1)} + i \rho_i^{(2)} \right) + \rho_i^{pc} \quad (9)$$

where the suffix  $i$  runs from 1 to 2.  $\rho^{pc}$  is a short notation for the contributions coming from the power corrections. It is not difficult to choose specific observable which are more sensitive to these complex parts than to the real ones. For example, simple counting arguments are sufficient to show that the difference

$$\sigma \equiv \sigma(\gamma \gamma \rightarrow \pi\pi) - \sigma(\gamma_R \gamma_R \rightarrow \pi^+ \pi^-) \quad (10)$$

is dominated by the lowest order spectral density and by the purely imaginary parts of their radiative corrections ( $\rho_\sigma \approx \rho_1^{(0)} \rho_2^{(0)} + \rho_1^{(2)} \rho_2^{(2)}$ ). As we are going to show in the next few sections, the imaginary parts of the sum rules of meson photoproduction can be classified according to a well defined factorized structure and are calculable in a close form.

Notice that the sum rule studied so far have been calculated in the approximation of massless quarks in the expansion of the correlator. Although this is clearly appropriate in the case of the pion, may not be appropriate for the investigation of heavier mesons.

In fact, the method can be easily applied to all the other reactions obtained by simple crossing (in the  $s$ ,  $t$  and  $u$  channels) of the Compton scattering amplitude, and in particular,

to processes of the form  $\gamma + \gamma \rightarrow A + \bar{A}$ , where  $A$  can be, in principle, any meson.

The evaluation of the sum rule, for these more general reactions, encounters a special difficulty related to the presence of a mass  $m$  in the diagrammatic expansion. As we are going to point out in the next few sections, this problem does not show up in the case of the form factor since the spectral density - at least to lowest order - is mass independent [5]. Such issues, although of technical nature, deserve special attention in the sum rule context. Our discussion, here, is ground-breaking since there are no available methods in the calculation of the OPE of the double spectral densities beyond 2- and 3-point functions. In particular we point out that the use of the Borel transform [1] in the isolation of such spectral functions must be handled with a certain care when the rank of the correlator increases (such as for 2 photon processes). We refer to the discussion in Appendix C for a presentation of the Borel method and its partial generalization to the evaluation of the lowest order spectral functions for Compton scattering.

In the case of the form factor, the radiative corrections, even in the massive case, can be obtained by the Borel method [10] which is, in fact, the easiest way to evaluate such contributions. As we are going to discuss next, in Compton scattering, these methods, [1] which rely on simple properties of the Borel transform and on the existence of unbound dispersion relations for the corresponding Feynman diagrams, are of limited use. Although the particular kinematics chosen in ref. [2] for the analysis of this reaction by sum rules (the requirement of treating on-shell physical photons) brings our treatment closer to the one in the form factor case, yet new analytical features emerge in the leading order spectral densities of these processes and in their power corrections [3]. Therefore, we proceed in sections 4 and 5 to develop our own approach to the identification of the complex contributions to the sum rules.

Our discussion, in the next sections, relies only on the analysis of massless and massive *scalar* amplitudes. There is no need, in fact, to consider the full expression of the relevant diagrams for processes of these type (and this is true also in the form factor) since the basic difficulties in the analytic properties of the spectral densities are all contained in the scalar amplitudes.

### 3 Massive correlators

Here we start our discussion of the role of a mass dependence by considering 3-point functions. The general features of the sum rule method for the analysis of form factors can be found in refs. [1] and [5]. In this case [5], the region of analyticity (for any 3-current correlator which interpolates with a specific form factor) is unbound and the dispersion integral extends up to infinity. In fact the basic spectral density is obtained from the integral (the *triangle* singularity) [11, 5]

$$\Delta_3(s_1, s_2, t) = \int d^4k \delta_+(k^2 - m_1^2) \delta_+((p_1 - k)^2 - m_2^2) \delta_+((p_1 - k + q_1)^2 - m_3^2), \quad (11)$$

The propagators of the cut lines have been replaced, as usual, by "positive" delta functions

$$\frac{1}{k^2 - m^2} \rightarrow -2\pi i \delta_+(k^2 - m^2), \quad (12)$$

as prescribed by Cutkosky rules. The relevant diagram is depicted in Fig. 1b. It gives the leading spectral function, for form factors, to lowest order in  $\alpha_s$  [5]. The dashed lines describe the usual cutting rules for the propagators. The spectral density for such a diagram takes the form (with  $p_1^2 = s_1$ ,  $p_2^2 = s_2$ )

$$\Delta_3(s_1, s_2, t) = \frac{\pi \theta(s_1 - 4m^2) \theta(s_2 - 4m^2)}{2 (\lambda(s_1, s_2, t))^{1/2}} \quad (13)$$

where

$$\lambda(s_1, s_2, t) = \left( (s_1 + s_2 - t)^2 - 4s_1 s_2 \right)^{1/2}. \quad (14)$$

is the Mandelstam function. The evaluation of (11) can be easily carried out in the Breit frame of the two (pion) lines characterized by the virtualities  $p_1^2$  and  $p_2^2$ . It is also easy to realize that (13) does not develop any additional singularity for positive  $s_1$ ,  $s_2$ , and negative  $t$  ( $t = (p_2 - p_1)^2$  fixed).

Another important observation, in the case of form factors, is the independence of the discontinuity eq. (11) from any mass in the propagator (see also [11]). Eq. (11) is in fact regular for any  $s_1$  and  $s_2$  over the entire complex plane of each of these variables. This allows us to write down a double dispersive representation of the massless triangle diagram in a rather straightforward way as simply as

$$\begin{aligned} T_3(p_1^2, p_2^2, t) &= \int \frac{d^4 k}{k^2 (p_1 - k)^2 (p_2 - k)^2} \\ &= \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\Delta_3(s_1, s_2, t)}{(s_1 - p_1^2)(s_2 - p_2^2)} + \text{subtr.} \end{aligned} \quad (15)$$

In eq. (15) we have omitted subtractions and single dispersive contributions to this integral. These two features 1) mass independence of the double spectral function and 2) its regularity over the entire  $s_1$  and  $s_2$  complex planes, are not preserved in Compton scattering, except for the massless case. Additional care is therefore required when we switch on a mass dependence in the basic diagrams, since new thresholds related to the  $u$ -cut in the dispersion variables automatically appears in the complex planes of the two virtualities  $s_1$  and  $s_2$ .

To be specific, let's consider the full contribution to the box diagram with a momentum flow chosen as in Fig. 1b, and for simplicity, let's restrict our considerations to the scalar case. We consider the following 4-point function

$$T_4 = \int \frac{d^4 k}{k^2 (p_1 - k)^2 (p_1 - k + q_1)^2 (p_2 - k)^2} \quad (16)$$



with massless propagators. At fixed angle and with

$$s + t + u = p_1^2 + p_2^2 \quad s > 0, \quad t < 0, \quad u < 0 \quad (17)$$

and moderately large  $s, t, u$ , assuming the existence of a region of analyticity of moderate size in  $s_1$  and  $s_2$  (see Fig. 2), we can write down a spectral representation for such integral of the form

$$T_4 = -\frac{1}{4\pi^2} \int_{\gamma_1} ds_1 \int_{\gamma_2} ds_2 \frac{\Delta(s_1, s_2, s, t)}{(s_1 - p_1^2)(s_2 - p_2^2)} \quad (18)$$

where the contours  $\gamma_{1,2}$ , both of a radius  $\lambda^2$ , are again chosen as in Fig. 2. If we introduce a double spectral function  $\Delta(s_1, s_2, s, t)$ , we can rewrite  $T_4$  as

$$T_4(p_1^2, p_2^2, s, t) = -\frac{1}{4\pi^2} \int_0^{\lambda^2} ds_1 \int_0^{\lambda^2} ds_2 \frac{\Delta(s_1, s_2, s, t)}{(s_1 - p_1^2)(s_2 - p_2^2)} + \dots \quad (19)$$

where the neglected pieces involve a complex part of the contour (the *background* contribution).

It has been shown in ref. [2] that the leading perturbative spectral function can be obtained, for this diagram, with the conditions on  $s$  and  $t$  given by (17), by the 3-cut integral (see Fig. 1b)

$$J(p_1^2, p_2^2, s, t, m = 0) = \int d^4k \frac{\delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2)}{(p_1 - k + q_1)^2}. \quad (20)$$

The evaluation of (20) has also been discussed in [2] and the answer, for the associated spectral density, turns out to be rather simple

$$\begin{aligned} \Delta(s_1, s_2, s, t) &= (-2\pi i)^3 J(p_1^2, p_2^2, s, t) \\ &= -\frac{4i\pi^4}{st}. \end{aligned} \quad (21)$$

Compared to the form factor case (see eq. (13)), the discontinuity along the cut,  $\Delta(s_1, s_2, s, t)$ , as given by (21) is  $s_1$  and  $s_2$  independent. This results seems to be oversimplified. However such a simplification is due to the fact that 1) we have neglected the quark masses in eq. (16) and 2) we have not considered other subleading cuts in the in the evaluation of (21) which are suppressed by power of  $s, t$  or  $u$ , compared to the leading result (the single dispersive contribution).

As we are going to show next, even in the simpler conditions of a kinematics of Compton type, a dispersion relation in  $s_1$  and  $s_2$  has unobvious features, not evident from eq. (21).

In fact, while for a vertex function there are only 3 kind of cuts and 3 independent variables (say  $s_1$ ,  $s_2$  and  $t$ ), one can vary any of them and keep the others below their thresholds, obtaining a dispersion integral extended up to infinity, in the case of the box diagram we have 7 cuts (respectively in  $s, t, u$ , plus the 4 external virtualities) and only 6 independent variables. The maximum number of variables which can be kept fixed is five.

A variation in the 6-th variable will then automatically affect the 7-th one. In our case, in particular, only 5 variables play a role since 2 external lines are fixed to be massless (photons of  $q_1^2 = q_2^2 = 0$ ) from the very beginning.

Let's suppose that we fix  $s$ ,  $t$  and  $s_2$  and vary  $s_1$ . Then the 5-th variable  $u = s_1 + s_2 - s - t$  can cross the threshold ( $u = 0$ ) (see Fig. 2), for a sufficiently large  $s_1$ , and become positive.

The diagram corresponding to this threshold is shown in Fig. 3. For  $m = 0$  and at fixed angle, however, such discontinuity doesn't play any role, as far as  $t$  is negative.

The proof of this last statement goes as follows. In Fig. 3 (for vanishing  $m$ ) the only possibility of having lines 1, 2 and 3 on shell at the same time, giving a non zero contribution to this diagram, is when these lines are collinear. For the same reason, lines 3, 4 and 5 must be collinear too. Therefore 1 and 5 must be collinear (i.e.  $q_1 \cdot q_2 = 0$ ), which implies that we must restrict ourselves to consider forward scattering from the very beginning. The requirement of working at fixed (and non forward or backward) angle ( $-t/s$  fixed) allows us to exclude such configurations.

A rapid look at the leading spectral density given by eq. (21) shows that this is indeed consistent with the expression that we obtained before for  $\Delta(s_1, s_2, s, t)$ , since at  $t = 0$  our spectral density becomes singular.

Therefore, as far as  $t$  remains in the physical region (and  $m = 0$ ), the contour representation of the box diagram eq. (18) (Fig. 2) can be sent to infinity and the Cauchy integral replaced by its double discontinuity eq. (21), integrated over the positive semiaxis of  $s_1$  and  $s_2$

$$T_4 = -\frac{1}{4\pi^2} \int_0^\infty \int_0^\infty ds_1 ds_2 \frac{\Delta(s_1, s_2, s, t)}{(s_1 - p_1^2)(s_2 - p_2^2)} + \text{subtr.} \quad (22)$$

This unexpectedly simple result is a consequence of the kinematics chosen in the derivation of the sum rule. This result, as we are going to see, is no longer valid when an explicit mass dependence in the diagrams is introduced.

Therefore, in the case  $m = 0$  and with on shell photons, it is also easy to write down the explicit expression of the dispersion integral in eq. (22) as simply as

$$T(p_1^2, p_2^2, s, t, \mu) = \frac{1}{2st(p_1^2 - \mu^2)(p_2^2 - \mu^2)} \text{Log} \left( \frac{p_1^2 - \mu^2}{p_2^2 - \mu^2} \right) + \text{subtr.}, \quad (23)$$

where we have chosen the same subtraction point  $\mu$  for both momenta  $p_1$  and  $p_2$  and neglected single dispersive parts in the integral. It is important to observe that the validity of a dispersion relation of the form given by eq. (22) is crucial in order to proceed to the evaluation of the spectral density by Borel techniques.

In order to illustrate how a bound on the validity of the dispersion relation comes into play, we have to reconsider the lowest order spectral density in its more general form, this time with a non zero mass in it. Below we are going to prove 3 basic facts:

1) for a non zero mass  $m$  the spectral density develops special singularities in the complex  $s_1$  and  $s_2$  planes;

2) the manifold spanned by the momenta which make the spectral density singular is a Landau surface for a  $u$ -channel singularity;

3) in the limit of vanishing mass the surface is degenerate and the spectral density remains regular, as shown above in eq. (21).

Let's proceed from point 1) and let's first discuss how a mass term is going to bound the region of analyticity.

The leading spectral function - for a non vanishing  $m$  - is obtained, in this more general case, from the 3-particle cut integral

$$J(p_1^2, p_2^2, s, t, m) = \int d^4k \frac{\delta_+(k^2 - m^2) \delta_+((p_1 - k)^2 - m^2) \delta_+((p_2 - k)^2 - m^2)}{(p_1 - k + q_1)^2 - m^2}, \quad (24)$$

and it can be expressed in the form

$$\begin{aligned} \Delta(s_1, s_2, s, t, m) &= (-2\pi i)^3 J(s_1, s_2, s, t, m) \\ &= - \frac{4\pi^4 i}{\sqrt{-t(4m^2 s_1 s_2 - 4m^2 s s_2 - 4m^2 s s_1 + 4m^2 s^2 + 4m^2 s t - s^2 t)}}, \end{aligned} \quad (25)$$

which reproduces eq. (21) in the  $m = 0$  case.

The evaluation of (25) is not obvious and can be performed following the steps discussed in Appendix A, where we discuss a similar integral. For two different masses, say  $m_1$  and  $m_2$ , in the box diagram, an expression of  $\Delta(s_1, s_2, s, t, m_1, m_2)$  in terms of  $s$  and  $t$ , as nice as eq. (25), is very difficult to obtain. In general, rational powers of the Mandelstam function  $\lambda(s_1, s_2, t)$  will appear and cannot be eliminated by a simple reshuffling of the independent variables  $s_1, s_2, s$ , and  $t$ .

Differently from eq. (21) which is mass independent and globally a regular function of the two pion' virtualities, eq. (25) shows, from its denominator, that the spectral function develops a singularity at  $s_2$  dependent positions in the  $s_1$  plane (and *viceversa*), as it is expected in general.

We intend to elaborate some more on the origin of this singularity which is of  $u$ -type, since it is generated when all the 4 internal lines in the massive box diagram go on shell (see the discussion in [9]).

We remind here that the conditions  $q_1^2, q_2^2 > 0$  has to be satisfied, in general, in order to get contribution from such discontinuity (for the box diagram), since a massless photon cannot decay into two massless particles at  $t < 0$ . The fact that such singularity reappears, also in the approximation of prompt photon emission ( $q_1^2 = q_2^2 = 0$ ) imposed from the very first stage of our calculations, shows that dispersion relations in two variables, for 4-point functions, are very difficult to handle on a very general base even for the simple box diagram.

Nevertheless we can easily prove that  $\Delta(s_1, s_2, s, t, m)$ , as given by eq. (25), becomes singular when the invariants  $s_1$  and  $s_2$  satisfy the Landau equation for obtaining a singularity of  $u$ -type.

The proof goes as follows. The leading singularity of the box diagram can be described geometrically by a simple dual diagram (generated by the momenta of the original box diagram)

[11] whose volume is imposed to be zero. This condition can be usually expressed in a more simple form in terms of some canonical variables  $y_{i,j}$ , proportional to the scalar product of the internal momenta in the diagram. In our case (for equal internal masses  $m$ ) such variables are

$$\begin{aligned} y_{12} &= -1 & y_{13} &= \frac{t - 4m^2}{2m^2} \\ y_{23} &= -1 & y_{14} &= \frac{s_1 - 2m^2}{2m^2} \\ y_{24} &= \frac{s - 2m^2}{2m^2} & y_{34} &= \frac{s_2 - 2m^2}{2m^2}. \end{aligned} \quad (26)$$

The equation of the Landau surface corresponding to Fig. 3 is in general given by a polynomial in  $y_{ij}$  [11]

$$\begin{aligned} &1 - y_{12}^2 - y_{13}^2 - y_{14}^2 - 2y_{12}y_{13}y_{23} \\ &- y_{23}^2 + y_{14}^2 y_{23}^2 - 2y_{12}y_{14}y_{24} - 2y_{13}y_{14}y_{23}y_{24} - y_{24}^2 + y_{13}^2 y_{24}^2 \\ &- 2y_{13}y_{14}y_{34} - 2y_{12}y_{14}y_{23}y_{34} - 2y_{12}y_{13}y_{24}y_{34} - 2y_{23}y_{24}y_{34} - y_{34}^2 + y_{12}^2 y_{34}^2 = 0 \end{aligned} \quad (27)$$

Inserting (26) into (27) one gets the expression

$$- \frac{t}{m^4} (4m^2 s_1 s_2 - 4m^2 s s_2 - 4m^2 s s_1 + 4m^2 s^2 + 4m^2 s t - s^2 t) = 0 \quad (28)$$

which clearly coincides with the singularity of the massive spectral density in eq. (25).

It is important to realize, however, that such singularity is far enough from the region of analyticity in which the dispersion relations are enforced.

In fact, the leading behaviour of such densities, even for non vanishing  $m$ , is still given (in eq. 25) by  $-1/(2st)$ , at large  $s$  and  $t$ , and at fixed value of their ratio  $-t/s$ . It's easy to show from the same equation that such a threshold is located at the typical values  $s_1, s_2 \approx s(1 + O(m^2/s))$  of the two virtualities, reasonably far away from the border of the contour of Fig. 2, for a dispersion relation in  $s_1$  and  $s_2$  to be valid.

Whence we can summarize our findings in this section as follows

1) We have explicitly described (by eq. (28)) the location of the  $u$  channel cut in the spectral density in the two complex planes of  $s_1$  and  $s_2$  and 2) we have shown that the only parameter which controls the position of this singularity is the mass in the correlation function.

At large  $s$  and  $t$  the term  $-s^2 t$  in eq. (28) *dominates* compared to all the other terms, even for values of the virtualities large compared to the mass  $m$  in the correlator, but smaller compared to  $s$  and  $t$ . The limit  $m \rightarrow 0$  can be studied from eq. (28) by sending to zero the mass in the numerator of such equation independently from the rest. The singularity surfaces degenerates, in this limit, the spectral density becomes regular and we reobtain eq. (21). In the next section we are going to extend our reasoning to the diagrams of the power corrections.

## 4 Power Corrections

In order to analyze the analyticity properties of the diagrams which appear in power corrections of gluonic type (see Fig. 4) we first review our method of calculation of such terms and show that in the massless case, again, the  $u = 0$  threshold, present in these diagrams, has a location in the complex planes of  $s_1$  and  $s_2$  which is parametrically controlled by the mass  $m$  of the OPE and disappears in the massless limit. We are then able to write down a dispersive representation of the diagrams for the power corrections which can be extended for  $s_1$  and  $s_2$  positive, ranging up to infinity. The pattern is similar to what already discussed in the previous section. Let's first observe that the power corrections can be obtained by the insertion of a modified quark propagator in momentum space [13] of the form

$$S(p) = \frac{\not{p} + m}{p^2 - m^2} + \frac{1}{2}i \frac{(\gamma^\alpha \not{p} \gamma^\beta G_{\alpha\beta} - m \gamma_\alpha G^{\alpha\beta} \gamma_\beta)}{(p^2 - m^2)^2} + \frac{\pi^2 \langle G^2 \rangle m \not{p}(m + \not{p})}{(p^2 - m^2)^4}, \quad (29)$$

which has quartic powers of momenta at the denominators.  $\langle G^2 \rangle$  is the gluonic condensate. Insertion of this propagator results in diagrams of the form shown in Fig. 4, for which we need an explicit spectral representation

$$J(p_1^2, p_2^2, s, t) \equiv \int \frac{d^4 k}{(p_1 - k)^2 (k^2)^2 ((p_2 - k)^2)^2 (p_1 - k + q_1)^2} \quad (30)$$

$$= -\frac{1}{4\pi^2} \int_0^{\lambda^2} ds_1 \int_0^{\lambda^2} ds_2 \frac{\bar{\Delta}(s_1, s_2, s, t)}{(s_1 - p_1^2)(s_2 - p_2^2)}. \quad (31)$$

We are interested in retrieving the explicit expression of  $\bar{\Delta}(s_1, s_2, s, t)$  for these diagrams.

For this purpose, as discussed in [3], it is convenient to introduce two auxiliary masses  $m_1$  and  $m_2$  and consider the more general spectral representation of the following amplitude

$$\begin{aligned} J(p_1^2, p_2^2, s, t, m_1, m_2) &= \int \frac{d^4 k}{(p_1 - k)^2 (k^2 - m_1^2)^2 ((p_2 - k)^2 - m_2^2)^2 (p_1 - k + q_1)^2} \\ &= \frac{\partial}{\partial m_1^2} \frac{\partial}{\partial m_2^2} \tilde{J}(p_1^2, p_2^2, s, t, m_1, m_2), \end{aligned} \quad (32)$$

where we have defined (see Fig. 4 a)

$$\tilde{J}(p_1^2, p_2^2, s, t, m_1, m_2) = \int \frac{d^4 k}{(k^2 - m_1^2) ((p_2 - k)^2 - m_2^2) (p_1 - k)^2 (p_1 - k + q_1)^2}. \quad (33)$$

and factorized two derivatives respect to the masses  $m_1$  and  $m_2$  in eq. (32).

The basic idea of the method used in [3] for the calculation of such corrections is to first approximate eq. (33) by its double dispersive part

$$\tilde{J}(p_1^2, p_2^2, s, t, m_1, m_2) = -\frac{1}{4\pi^2} \int_0^{\lambda^2} ds_1 \int_0^{\lambda^2} ds_2 \frac{\tilde{\Delta}(s_1, s_2, s, t, m_1, m_2)}{(s_1 - p_1^2)(s_2 - p_2^2)} \quad (34)$$

with a spectral density given by

$$\begin{aligned} & \Delta^{scalar}(s_1, s_2, s, t, m_1, m_2) \\ &= (-2\pi i)^3 \int d^4 k \frac{\delta_+(k^2 - m_1^2) \delta_+((p_2 - k)^2 - m_2^2) \delta_+((p_1 - k)^2)}{(p_1 - k + q_1^2)}. \end{aligned} \quad (35)$$

$\lambda^2$  is the radius of a finite contour in each plane of  $s_1$  and  $s_2$ . Then, the dispersive part of the scalar integral associated to eq. (31) is obtained by setting  $m_1$  and  $m_2$  to zero according to the relation

$$\bar{\Delta}(s_1, s_2, s, t) \equiv \left( \frac{\partial}{\partial m_1^2} \frac{\partial}{\partial m_2^2} \Delta^{scalar}(s_1, s_2, s, t, m_1, m_2) \right)_{m_1, m_2=0}. \quad (36)$$

Evaluation of (35) gives

$$\tilde{\Delta}(s_1, s_2, s, t, m_1, m_2) = (-2\pi i)^3 \frac{\pi}{2\delta W(m_1, m_2)} \quad (37)$$

with

$$\begin{aligned} W(m_1, m_2) = & -4m_1^2(Q^2)^2 + 2m_2^2 Q^2 s + 4(Q^2)^2 s + 2m_1^2 Q^2 s_1 - 2m_2^2 Q^2 s_1 \\ & -2Q^2 s s_1 + 2m_1^2 Q^2 s_2 - 2Q^2 s s_2 - m_1^2 s_1 s_2 + s s_1 s_2 \end{aligned} \quad (38)$$

$Q^2$  is given by

$$Q^2 = \frac{1}{4}(s_1 + s_2 - t + \lambda^{1/2}(s_1, s_2, t)) = \frac{1}{4}(s + u + \lambda^{1/2}(s_1, s_2, t)), \quad (39)$$

with

$$\begin{aligned} \lambda(s_1, s_2, t) &= s_1^2 + s_2^2 + t^2 - 2s_1 t - 2s_2 t - 2s_1 s_2 \\ &= \left( \frac{4Q^4 - s_1 s_2}{2Q^2} \right)^2. \end{aligned} \quad (40)$$

As described in Appendix A,  $Q$  is the large light-cone component in the "plus" direction of the incoming momentum  $p_1$ . Specifically, we expand all the momenta in the Sudakov base

$$p_1 = Qn^+ + \frac{s_1}{2Q}n^-, \quad p_2 = \frac{s_2}{2Q}n^+ + Qn^-, \quad (41)$$

where  $n^+$  and  $n^-$  are light-cone momenta, such that  $n^{+2} = n^{-2} = 0, n^+ \cdot n^- = 0$ . In our conventions

$$n^+ = \frac{1}{\sqrt{2}}(1, \mathbf{0}_\perp, 1), \quad n^- = \frac{1}{\sqrt{2}}(1, \mathbf{0}_\perp, -1). \quad (42)$$

The momentum transfer  $t$  can be expressed in terms of  $Q^2$ , which can be viewed approximately as a large "parameter" in the scattering process, by the relation

$$t = \frac{-((2Q^2 - s_1)(2Q^2 - s_2))}{2Q^2} \quad (43)$$

Notice that  $t = -2Q^2$  for  $s_1 = s_2 = 0$ .

Now, simple manipulation of eqs. (37) and eq. (36) give

$$\bar{\Delta}(s_1, s_2, s, t) = \frac{-32\pi^4 i (Q^2)^2 (s_1 - s) (-4(Q^2)^2 + 2Q^2 s_1 - s s_1 + 2Q^2 s_2)}{s^3 (2Q^2 - s_1)^3 (-2Q^2 + s_2)^3}. \quad (44)$$

We now want to demonstrate that

- 1) the only singularities which bound the dispersion integrals (33) for the power corrections are of  $u$ -type
- 2) they disappear in the massless limit.

For this purpose, let's build on the results of the previous section by showing that, even for diagrams with gluonic insertions (Fig. 4) the singularity surface of  $u$ -type coincides with the singularity surface of the spectral density eq. (37). In particular, all the considerations made in the previous section for the case of the leading order spectral density - regarding the size of the region of analyticity of the correlator and the position of the  $u$ -cut - remain true also in this more general case. It is also important to observe that this is consistent with the structure of eq. (32) since the two derivatives which have been factorized in that equation (for power corrections we deal with *quartic* propagators) do not change the location and the nature of the singularities of the integral given by eq. (33).

The discussion that follows also confirms the validity of the method developed in ref. [3] for the evaluation of the power corrections in Compton processes.

Specifically we are going to show that

- 1) the singularity of  $u$  type in the power correction diagram (Fig. 4 b) is equivalent to the vanishing of  $W(m_1, m_2)$  which is given in eq. (38), and appears in the denominator of the spectral density eq. (37) of the power corrections;
- 2) in the massless limit such singularity surface disappears and the spectral density (37) turns into the regular function (44).

Similarly to what already discussed in the previous section, let's define the parameters  $y_{ij}$  for this new diagram, as we have done in the previous section, which are helpful to describe such a singularity surface

$$y_{12} = -\frac{(m_3^2 + m_4^2)}{2m_3 m_4} \quad y_{13} = \frac{t - m_3^2 - m_2^2}{2m_3 m_2},$$

$$\begin{aligned}
y_{23} &= -\frac{m_4^2 + m_2^2}{2m_4m_2}, & y_{14} &= \frac{s_1 - m_3^2 - m_1^2}{2m_1m_3}, \\
y_{24} &= \frac{s - m_4^2 - m_2^2}{2m_2m_4}, & y_{34} &= \frac{s_2 - m_2^2 - m_1^2}{2m_1m_2}.
\end{aligned} \tag{45}$$

The equation defining the Landau surface for diagrams of box typ, even in the presence of quartic propagators, is still given by eq. (27). It is possible to rearrange the equation of such a surface in the form

$$\frac{P(Q^2, m_1, m_2, m_3, m_4)}{64m_1^2m_2^2m_3^2m_4^2Q^2} = 0 \tag{46}$$

where  $m_i$  is the mass of the  $i$ -th internal line, and  $P(Q^2, m_i)$  is a complicated polynomial in the masses  $m_i$  and in the momentum  $Q$ , whose expression is too lengthy to be given here. Although the calculations are quite involved, nevertheless it is possible to show that in the limit  $m_3, m_4 \rightarrow 0$  eq. (46) simplifies since

$$P(Q^2, m_1, m_2, m_3, m_4) \rightarrow W(m_1, m_2)^2 \tag{47}$$

and becomes

$$W(m_1, m_2)^2 = 0, \tag{48}$$

with  $W(m_1, m_2)$  given by eq. (38) and clearly coincides with the singularity surface of eq. (37).

Therefore we have shown that in the massive case - even for different masses - and, in particular, in the diagrams of the gluonic power correction, the spectral density develops singularities of  $u$ -type as conjectured in ref. [2]. Such singularities are absent for vanishing  $m_i$  and the spectral integral, in this specific case, can be extended to the entire positive axis of  $s_1$  and  $s_2$ . Therefore, we can extend the results of ref. [3] and write down a new dispersive representation of the power correction integral eq. (31) for  $s_1, s_2 > 0$  as

$$J(p_1^2, p_2^2, s, t) = -\frac{1}{4\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\bar{\Delta}(s_1, s_2, s, t)}{(s_1 - p_1^2)(s_2 - p_2^2)}. \tag{49}$$

where we have omitted, again, single dispersive contributions. A check on the consistency of our results can be obtained immediately since  $\bar{\Delta}(s_1, s_2, s, t)$  can be rewritten in the form

$$\bar{\Delta}(s_1, s_2, s, t) = \frac{32\pi^4 i (s_1 - s)(ss_1 - 2s_1s_2 - s_1t - s_2t + t^2 - 2t\lambda^{1/2}(s_1, s_2, s, t))}{(s_1 + s_2 - t + \lambda(s_1, s_2, s, t))s^3t^3}, \tag{50}$$

and the only singularity displayed by (50) is a pole at  $t = 0$ . For scattering at fixed angle this pole doesn't play any role.



## 5 Radiative Corrections

In this section we develop methods for the calculation of the radiative corrections to the lowest order sum rules. Although we are not going to discuss the complete evaluation of these terms to the actual case of pion photoproduction, which we leave as a future investigation, here we will describe methods which may have application to a large class of reactions of Compton type and are easily generalizable to the proton case. The method used is simply based on the observation that the complex contributions to the spectral densities can be "decomposed" in specific subdiagrams which involve 2- and 3-particle cuts *times* the lowest order result. After this observation, we are then able to treat these contributions in a systematic way. Unfortunately, from a practical viewpoint, general schemes to treat such corrections are not available, since double dispersion relations involve complex spectral densities beyond leading order. So, one has to resort on a case by case study. In particular, the treatment of the infrared divergences in the dispersion formalism, for simple dispersive integrals, and the problems which naturally come from its application, are well known [20]. Infrared divergences are often regulated by introducing a fictitious mass in the discontinuity function, as done, in some cases, for the form factor. In our case such a problem is even more complicated to deal with, since we are handling double dispersion integrals. As we have learned from the previous sections, a mass term *bounds* the spectral density and modifies its singularities. Particular care is therefore required when trying to take the massless limit in infrared sensitive integrals [20].

The method that we are going to illustrate in this section has two main features, specifically

- 1) it combines the good quality of the "Breit frame" (namely the fact that in the "Q"-variable the spectral density is polynomial) with
- 2) dimensional regularization.

These two properties are crucial in order to do the calculations in close form. Dimensional regularization, in particular, is used in a plane which is transversal with respect to the two (longitudinal) light-cone variables in the  $n^\pm$  directions. The difference of the method, compared to the lowest order calculation, appears only in the angular variables, since the angular integral has now dimension  $n - 3$ . Given the fact that the lowest order result spectral density is infrared safe, the use of dimensional regularization as  $n$  goes to 4, does not modify the lowest order expression of the spectral density [16] (see Appendix C).

General arguments regarding the infrared safety of the sum rule method to Compton scattering have been presented in ref. [2]. In a direct approach, however, one has to be able to show explicitly that the infrared divergences generated by the all possible cuts cancel. We reserve the discussion of this important point elsewhere. The treatment of the ultraviolet divergences, instead, is much less difficult since can be controlled by simple subtractions. In particular, for Borel sum rules, such difficulty is largely absent due to the fast convergence of the spectral densities after Borel transforms. This discussion can be addressed in a better way in the actual case of Compton scattering rather than in an auxiliary scalar theory, as we are doing it now. In particular, the use of Ward identities in a true gauge theory simplifies drastically the treatment and can possibly make such infrared cancellations more transparent.

Notice that standard tools, such as the largest time equation, commonly used in the calculation of single variable cuts in diagrams with any number of external lines, (see [8]) are of very limited help in the classification of the double spectral densities of Feynman integrals.

In order to make our discussion as clear as possible, we have decided to treat one specific diagram with mass regulators and the remaining ones by dimensional regularization. The treatment of this first diagram (which is, however, *real*) is quite involved, but it may serve illustrate quite well our general approach, and how it is possible to derive a set of basic rules according to which we can deduce whether a particular cut gives a *real* or a complex contribution to the OPE.

For this purpose let's consider Fig. 5a with the momentum (and energy) flow chosen as shown in this figure.

Let's first observe that the double imaginary part of such diagram (Fig. 5b) can be split into the convolution of a 2-particle discontinuity in the  $s'$  energy variable, specifically  $s' = (p_1 - l)^2$  and in a 2-to-2 scattering amplitude ( the top of the diagram), calculated at Born level, with on-shell external lines (Fig. 5c).

The 2-particle cut (bottom part of Fig. 5c) contributes to the total discontinuity through the sub-integral

$$\Delta^{5b}(s_1, s_2, p_1 \cdot l)$$

$$\int d^4k \frac{\delta_+(k^2 - m^2) \delta_+((p_1 - k + l)^2 - \mu^2)}{((p_1 - k)^2 - m^2)((p_2 - k)^2 - m^2)} = \frac{s'^2}{2\pi D^{1/2}} \text{Ln} \left( \frac{a c - b d \cos \theta + D^{1/2}}{a c - b d \cos \theta - D^{1/2}} \right) \quad (51)$$

where

$$\begin{aligned} a &= s'^2 - s'(s_1 + \mu^2 - m^2) - (\mu^2 - m^2)s_1; \\ b &= \lambda^{1/2}(s', \mu^2, m^2) \lambda^{1/2}(s', s_1, 0); \\ c &= s'^2 - s'(s_2 + \mu^2 - m^2) + (\mu^2 - m^2)s_2; \\ d &= \lambda^{1/2}(s', \mu^2, m^2) \lambda^{1/2}(0, s_2, s'); \\ q_{12} &= q_1 - q_2; \\ D &= (a c - b d \cos \theta)^2 - (a^2 - b^2)(c^2 - d^2); \\ \cos \theta &= s' / ((s' - s_1)(s' - s_2))(2t + s' - s_1 - s_2 - s_1 s_2 / s'); \\ s' &= s_1 - 2p_1^+ l^- - 2p_1^- l^+. \end{aligned} \quad (52)$$

and where  $\mu$  and  $m$  are regulator masses. They allow us to keep under control the divergences of these sub-cuts.

In order to obtain the full contribution of diagram 5b we have just to convolute its top part, shown in Fig. 5c, with (51), in the form

$$\int_{-\infty}^{\infty} dl^+ \int_{-\infty}^{\infty} dl^- \int d^2 l_{\perp} \frac{\Delta_{5b}(s_1, s_2, l) \delta_+(l^2 - m^2) \delta_+((q_{12} + l)^2 - m^2)}{(q_1 + l)^2 - m^2}. \quad (53)$$

The integrals over the transversal components of  $l$  and on one of the two longitudinal components, say  $l^-$ , can be done elementarily (since  $q_{12} \cdot l_{\perp} = 0$ ) while the  $l^+$  integrals is cut from above and from below by the conditions on the momentum flow. The analysis of these conditions is algebraically very involved and is briefly sketched in Appendix B.

It is possible to simplify eq. (53) even further.

For this purpose it is crucial to observe that, in the Breit frame, eq. (51) has no dependence on the angular variables of the transversal plane, which have been introduced by the Sudakov decomposition of the internal momentum  $l$

$$l = l^+ n^+ + l^- n^- + l_{\perp}. \quad (54)$$

After having performed the integration over the transversal variables, the final result for the diagram of Fig. 5b can be expressed in the form

$$I_{5b} = \frac{\pi}{4q_{12}^+} \int_{\mu_1}^{\mu_4} dl^+ \frac{\Delta_2(l^+, l_m)}{\left((q_1^+ l_m + q_1^- l^+)^2 - 2q_1^+ q_1^- (2l^+ l_m - m^2)\right)^{1/2}} \quad (55)$$

where

$$l_m \equiv -\frac{(q_{12}^2 + 2q_{12}^- l^+)}{2q_{12}^+}. \quad (56)$$

The explicit expressions of  $\mu_1$  and  $\mu_4$ , are given in Appendix B, together with a brief discussion.

The contribution from diagram 5b is, however, real - for very small values of the regulator masses  $m$  and  $\mu$  - and therefore cannot be responsible for any interference. In order to illustrate this last point, let's show first that the integrand in (55) is a regular function of  $l^+$  in the interval of integration. Our analysis is exact in the limit of  $m \rightarrow 0$ . An independent check of our reasoning will also be discussed below. Let's rewrite (56) as

$$l_m = \frac{A_1 l^+ + B_1}{(2Q(2Q^2 - s_2)(4Q^4 - s_1 s_2))^2}, \quad (57)$$

where

$$\begin{aligned} A_1 &= 2Q(2Q^2 - s_1)(4Q^4 - s_1 s_2)^2, \\ B_1 &= (-2Q^2 + s_1)(-2Q^2 + s_2)(-16Q^8 + 16Q^6 s - 8Q^4 s^2 + 4Q^2 s s_1 s_2 - s_1^2 s_2^2). \end{aligned} \quad (58)$$

Inserting this result in the denominator of eq. (55) we obtain

$$(q_1^+ l_m + q_1^- l^+)^2 - 2q_1^+ q_1^- (2l^+ l_m - m^2) = \frac{N_1 l^{+2} + N_2 l^+ + N_3}{4Q^2(4Q^4 - s_1 s_2)^6}, \quad (59)$$

where

$$\begin{aligned} N_1 &= (2Q^2 - s_1)(4Q^4 - s_1 s_2)^6 \\ N_2 &= 2Q(s - 2Q^2)(2Q^2 - s_1)^2(2Q^2 - s_2)(4Q^4 - s_1 s_2)^3 \\ &\quad \times (16Q^8 - 16Q^6 s + 8Q^4 s^2 - 4Q^2 s s_1 s_2 + s_1^2 s_2^2) \\ N_3 &= m^2 4Q^2 (s - 2Q^2)(-2q^2 + s_1)(-2Q^2 + s_2)(4Q^4 - s_1 s_2)^4 (2Q^2 s - s_1 s_2). \end{aligned} \quad (60)$$

In the limit  $m \rightarrow 0$  the quadratic form (59) has only complex roots, since its discriminant vanishes. Therefore there are no singularities of the integrand of (55) in the variable  $l^+$ . A small finite mass term does not modify our conclusions, since it is possible to show, even in this case, that the two (now real) roots still lay outside the integration region in  $l^+$ .

An alternative, more direct proof of this fact can be obtained in the following way. It is easy to realize that any (eventual) additional imaginary part in (55) is generated when the residual propagator of momentum  $p_1 + l$  (see also eq. (53)) goes on-shell. The discussion outlined in section 3 shows that this is indeed not possible in the limit of a small mass  $m$ , unless  $t = 0$ . Therefore the integrand is regular.

After these considerations, we are allowed to do a simple counting of the complex factors of  $i$ , for this specific diagram, as a check of our conclusions: we insert of a factor  $i^4$  (from the 4 single-particle cuts) times a factor of  $i^2$  from the two independent loop integrations to decide of the overall factor. The result is therefore real.

Other imaginary parts are generated by diagrams of the form 6a and 6c, in which the insertion of radiative corrections of vertex-type and self-energy are considered.

Let's now analyze Fig. 5d. In this case - and we shall proceed in a similar way in the other similar cases - we prefer to use dimensional regularization to evaluate the sub-cuts which appear in the relevant Feynman diagrams. The divergences, in fact, emerge at this level. The contribution of these sub-diagrams are then folded with a 3-particle cut integral which is further evaluated by using Sudakov variables in the Breit frame of the two pion' lines.

The method is particularly convenient in order to check the infrared safety of the final sum of all the diagrams.

For this purpose let's decompose 5d as shown in 5e, where the top part now involves a 2-particle cut with 4 on-shell external lines.  $s' = (p_1 - k + q_1)^2$ , in this particular case, denotes the invariant energy in the "s" channel of this subdiagram. In  $n = 4$  spacetime dimensions this 2-particle cut is infrared divergent. The situation is similar to the case discussed above (diagram 5b), when the two masses of this diagram were set to zero. Notice that, in principle, also 5b can be regulated dimensionally, although the expression of its 2-particle contribution is far less obvious, since it involves an imaginary part with 2 off-shell external lines.

Let's then define the  $n$ -dimensional expression of this 2-particle cut contribution when all the external lines are massless

$$\Delta_{box}(s') = \int d^n k \frac{\delta_+(k^2) \delta_+((p_1 - k + q_1)^2)}{(2\pi)^4 (p_1 - k)^2 (p_2 - k)^2}. \quad (61)$$

There exist various possible way to evaluate (61), the simplest probably being the one discussed in [15]. This same method can be simply used to regulate by dimensional regularization also eq. 55. In [15], (61) is calculated by the method of Feynman parameters combined with the Kummer series. A simpler way to obtain this result is, as usual, to simplify the constraints imposed by the Dirac's delta function in a suitable frame. The derivation of these discontinuities is quite straightforward and we omit any of the details and quote the final result. We get, for the box diagram (top of Fig. 5e)

$$\begin{aligned} \Delta_{box}(s') &= \frac{\pi^{n/2-1}}{4} \frac{\Gamma[\frac{n}{2} - 2]^2}{\Gamma[n/2 - 1] \Gamma[n - 4]} \\ &\quad \times F(1, 1, \frac{n}{2} - 1, 1 + \frac{s'}{t}) s'^{n/2-4}, \end{aligned} \quad (62)$$

where  $F(a, b, c, z)$  denotes, as usual, the hypergeometric function [18]. The 2-particle cut for the vertex function (Fig. 6b) is instead given by

$$\begin{aligned} \Delta_{vertex} &= \int d^n k \frac{\delta_+(k^2) \delta_+((p_1 - k)^2)}{(k - p_1)^2} \\ &= -\frac{\pi^{n/2-1}}{4} \frac{\Gamma[n/2 - 2]}{\Gamma[n - 3]} s'^{n/2-3}. \end{aligned} \quad (63)$$

The expansion around  $n = 4 + \epsilon$  of eq. (62) generates a single pole in  $1/\epsilon$  times a combination of polylogarithmic functions, the last ones obtained from the Taylor expansion of the hypergeometric function

$$F(1, 1, 1 + \frac{\epsilon}{2}, 1 + \frac{t}{s'}) = \left(\frac{-t}{s'}\right)^{\epsilon/2-1} \left(1 + \frac{1}{4}\epsilon^2 Li_2(x) + \frac{1}{8}\epsilon^3 (S_{1,2}(x) - Li_3(x)) + O(\epsilon^4)\right), \quad (64)$$

where  $x = 1 + t/s'$ .

The definitions of  $Li_2(x)$  and  $S_{1,2}$  are given in ref.[20]. We can easily convolute this result with the expression of the 3-particle cut given in eq. (20) to obtain

$$\begin{aligned} \Delta_{5d} &= \int d^n k \delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2) \Delta_{box}((p_1 - k + q_1)^2) \\ &= \Gamma_1 \int d^n k \delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2) \\ &\quad \times F\left(1, 1, n/2 - 1, 1 + t/(p_1 - k + q_1)^2\right) \left((p_1 - k + q_1)^2\right)^{n/2-4} \end{aligned}$$

where we have set

$$\Gamma_1 \equiv \frac{\pi^{n/2-1}}{2} \frac{\Gamma[n/2-2]^2}{\Gamma[n/2-1]\Gamma[n-4]}. \quad (65)$$

Notice that all the infrared sensitive parts in (65) are only contained in the factor  $\Gamma_1$ . This is a nice feature of Dimensional Regularization, which allows us to evaluate completely eq. (65) till the last stage. In fact, defining  $\epsilon = n - 4$ , the pole contribution and the finite parts of (65) can be obtained from the relation

$$\Delta_{5d} = \mu^\epsilon \left(-\frac{\pi}{t}\right) \left(\frac{-t}{\mu^2}\right)^{-\epsilon/2} \Gamma_1 \int d^n k \frac{\delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2)}{(p_1 - k + q_1)^2}, \quad (66)$$

where we have introduced a renormalization mass scale  $\mu$ . Notice that it is necessary to perform also the remaining  $k$  integral in  $4 + \epsilon$  dimensions, since it contributes to the finite part. The basic integral which appears here is

$$\sigma_0 = \int d^n k \delta_+(k^2) \frac{\delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2)}{(p_1 - k + q_1)^2}, \quad (67)$$

$$(68)$$

whose evaluation is similar to the one discussed in Appendix C. Expanding in  $\epsilon$  the logarithmic part of (66) we get

$$\Delta^{5d} = \mu^\epsilon \Gamma_1 \sigma_0 (1 - \epsilon/2 \text{Log}[\frac{-t}{\mu^2}]). \quad (69)$$

Inserting the explicit expression of  $\sigma_0$  we finally obtain

$$\begin{aligned} \Delta^{5d} &= \mu^\epsilon 2^{n-5} \pi^{n-5/2} \frac{\Gamma[n/2-2]^2 \Gamma[n/2-3/2]}{\Gamma[n/2-1] \Gamma[n-4] \Gamma[n-3]} F(1/2, 3/2, n/2-1, v) \\ &\times \frac{J(Q, s_1, s_2)}{A} \left(1 - \frac{\epsilon}{2} \text{Log}[\frac{-t}{\mu^2}]\right). \end{aligned} \quad (70)$$

We have defined

$$\begin{aligned} A &= s - s_1 - 2k^+ q_1^- - 2k^- q_1^+ \\ &= \frac{(2Q^2 - s_1)(2Q^2 - s_2)(4Q^4 s - 4Q^2 s_1 s_2 + s s_1 s_2)}{(4Q^4 - s_1 s_2)}, \end{aligned} \quad (71)$$

$$\begin{aligned} B &= 2|k_\perp||q_\perp| \\ &= \left(2Q^2 s_1 s_2 (s - 2Q^2)(2Q^2 s - s_1 s_2)\right)^{1/2} \\ &\times \frac{(2Q^2 - s_1)(2Q^2 - s_2)}{(4Q^4 - s_1 s_2)}, \end{aligned} \quad (72)$$

and set  $v \equiv B/A$ . Notice that  $A > 0$ , and  $B > 0$ ). Notice also that the momenta  $k^\pm$  are fixed by the (lowest order) 3-particle cut (Fig. 1b) in terms of  $Q$ ,  $s$  and of the two virtualities  $s_1$  and  $s_2$ , and that  $v \approx 1/Q^4 \ll 1$ , so an expansion up to order  $v^2$  is a very good approximation to the integrals above.

The evaluation of 6a proceeds in a similar way, though it is slightly more complex.

In this case we get

$$\Delta^{6a} = \Gamma_2 \int d^n k \frac{\delta_+(k^2)\delta_+((p_1 - k)^2)\delta_+(p_2 - k)^2}{((p_1 - k + q_1)^2)} \Delta_{vertex}((p_1 - k + q_1)^2). \quad (73)$$

Using the explicit expression of eq. (63) we get

$$\Delta^{6a} = \Gamma_2 \int d^n k \frac{\delta_+(k^2)\delta_+((p_1 - k)^2)\delta_+(p_2 - k)^2}{((p_1 - k + q_1)^2)^2} \left( 1 - \frac{\epsilon}{2} \ln \left( \frac{(p_1 - k + q_1)^2}{\mu^2} \right) \right). \quad (74)$$

We have defined

$$\Gamma_2 = -\frac{\pi^{n/2-1}}{2} \frac{\Gamma[n/2 - 2]}{\Gamma[n - 3]}. \quad (75)$$

From the  $k$  integral we obtain two main terms:

$$\sigma_1 = \int d^n k \frac{\delta_+(k^2)\delta_+((p_1 - k)^2)\delta_+(p_2 - k)^2}{((p_1 - k + q_1)^2)^2} \quad (76)$$

and the logarithmic contribution

$$\sigma_2 = \int d^4 k \frac{\delta_+(k^2)\delta_+((p_1 - k)^2)\delta_+(p_2 - k)^2}{(p_1 - k + q_1)^2} \ln \left( \frac{(p_1 - k + q_1)^2}{\mu^2} \right). \quad (77)$$

Their evaluation is discussed in Appendix C and their contribution is finite. The single pole in  $1/\epsilon$ , in diagrams of this type, is generated only by the 2-particle cut of the vertex function shown in Fig. 6b

$$\begin{aligned} \Delta^{6a} = & \frac{\mu^\epsilon}{A^2} J(Q, s_1, s_2) \pi^{n-5/2} \frac{\Gamma[n/2 - 2]}{\Gamma[n - 3]\Gamma[n/2 - 3/2]} \\ & \times \left( 2^{n-5} \frac{\Gamma[n/2 - 3/2]^2}{\Gamma[n - 3]} F(1, 3/2, n/2 - 1, v) (1 - \text{Log}[\frac{(1+v)A}{\mu^2}]) \right. \\ & \left. - \pi \epsilon \frac{v}{1+v} \right) \end{aligned} \quad (78)$$

Again, in practical applications, just the first few terms in  $v$  are necessary.

A radiative correction of another type, also responsible of the generation of imaginary parts is shown in Fig. (6c), and it corresponds to a self energy insertion on the fermionic line at the top of the box diagram. An odd number of particle cuts is involved in this diagram. As we have discussed in the former cases, also here, again, we can unfold the 5-particle cut into a 2- times a 3-particle cut. Both cuts are regular in  $n = 4$  dimension, so there is no need to use dimensional regularization. For the 2-particle cut we get ( $q$  is the external momentum)

$$\int d^4k \delta_+(k^2) \delta_+((q-k)^2) = \frac{\pi}{2} \theta(q^2), \quad (79)$$

and the 5-particle cut can then be set in the form

$$\int d^n k \frac{\delta_+(k^2) \delta_+((p_1-k)^2) \delta_+((p_2-k)^2) \theta((p_1-k+q_1)^2)}{((p_1-k+q_1)^2)}. \quad (80)$$

It is not so difficult to show that the condition  $(p_1 - k + q_1)^2 > 0$  is identically satisfied for all possible values of  $k$  in the integral.

Evaluating this integral at  $n = 4$  we get

$$\Delta^{6d} = -\frac{\pi}{2stA(1-v^2)}. \quad (81)$$

The evaluations presented in this section, as we have seen, can be carried on up to the last stage. This is a remarkable feature. A simple factorized structure for the imaginary parts generated by the radiative corrections to the lowest order sum rules emerges. All these corrections, (see Fig. 7), can be easily evaluated by these methods.

It is not difficult to realize that the set displayed in Fig. 7 is the complete one. The diagrams shown in Fig. 8, for instance, have not been added to the list. Let's see why.

As an example, let's consider the diagram shown in Fig. 8a. From a first inspection, it seems that this diagram should be included to the list since the number of its cuts is odd. However, it is easy to realize that it involves a subcut of  $t$ -type, which is non vanishing only if  $t$  is in the unphysical region ( $t > 0$ ). Therefore 8a can not contribute as far as  $t$  remains in the physics region.

Other contribution to the real part of the spectral density are displayed in Figs. 8b and 8c. Each of these two diagrams has a maximal number of internal on-shell lines and no other additional line can go on shell. In fact any other additional line, if on-shell, induces at least one vertex with 3 on-shell massless lines in the new diagram. The presence of any vertex of this type then, will force all the 3 particles at that vertex to be collinear. This condition is, in general, too strong to give a non vanishing discontinuity, except at specific points in the phase space of the process. The discussion presented in section 3 may serve well illustrate this aspects. In fact, in that case,  $t$  was forced to be zero in the presence of such configurations in the reduced diagram.



## 6 Conclusions

We have discussed in detail some crucial aspects concerning the analyticity properties of the spectral densities in Compton processes. The analytical bounds on the region in which a dispersion relation is valid is controlled by the quark mass in the expansion of the correlator.

For nonvanishing  $m$  the spectral density has special singularities, absent in the form factor case, which have been identified as Landau surface of  $u$ -type. At order  $\alpha_s$  they are important in the analysis of the radiative corrections.

As we have seen, in the massless case, a simplified picture of such corrections emerges. Remarkably, the evaluation of the complex parts, responsible of effect of interference in the sum rules, can be done in close form. For this purpose, Sudakov methods in  $n$  dimensions and dimensional regularizations have been developed and combined in an original way. Since it is generally believed that, at intermediate energy, the phases of elastic scatterings are of non-perturbative origins, it is interesting to see how relevant the "Feynman mechanism" [1] which motivates the sum rule description is - compared to hard scattering factorization - in the description of such effects.

We reserve to discuss the phenomenological application of our results to the evaluation of the phases of pion photoproduction elsewhere[17].

However, from this work, a well defined structure for the complex spectral densities for  $\gamma$  collisions emerges, which can be enforced in an actual calculation.

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## Appendix A. The light-cone formulation.

Here we summarize briefly the kinematics of Compton processes and point out the importance to work out the expression for the spectral densities in the light-cone frame since, in such a frame, they become polynomial. The simplification is enormous since the Mandelstam function (see eq. 11) -which appears in the form factor case- is absorbed into the definition of a large "plus" light-cone momentum of one of the off shell parton' lines. It is important to notice that, since from the very beginning we have been working with a diagrammatic expansion of current correlators, in each diagram in the O.P.E. (see Fig. 1) the virtuality of the pion' lines will, in general, be different.

Nevertheless we can picture Fig. 1 as an ordinary scattering process and select a special frame in which all the transversal momentum belongs to just a single particle (the photons). Let's define

$$s = (p_1 + q_1)^2 \quad t = (q_2 - q_1)^2, u = (p_2 - q_1)^2, \quad (\text{A.1})$$

with

$$s + t + u = s_1 + s_2. \quad (\text{A.2})$$

We also expand the momenta of the incoming photon as

$$q_1 = q_1^+ n^+ + q_1^- n^- + q_{1\perp}. \quad (\text{A.3})$$

In the Breit frame of the incoming meson we have

$$u = (p_2 - q_1)^2 = 2Q^2 - s + \frac{s_1 s_2}{2Q^2}. \quad (\text{A.4})$$

Covariant expressions for  $q_1^\pm$  and  $q_2^\pm$  can be easily obtained in the form

$$q_1^+ = \frac{(s - 2Q^2)(2Q^2 - s_2)}{2Q\delta}, \quad (\text{A.5})$$

$$q_1^- = \frac{(2Q^2 - s_1)(2Q^2 s - s_1 s_2)}{4Q^3 \delta}. \quad (\text{A.6})$$

In the light cone frame the spectral function can be calculated to lowest order in terms of 3-cut integrals of the form [2] [3]

$$I[f(k^2, k \cdot p_1, \dots)] = \int d^4 k f(k^2, k \cdot p_1, \dots) \frac{\delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2)}{(p_1 - k + q_1)^2}, \quad (\text{A.7})$$

$$I'[f(k^2, k \cdot p_1, \dots)] = \int d^4 k f(k^2, k \cdot p_1, \dots) \delta_+(k^2) \delta_+((p_1 - k)^2) \delta_+((p_2 - k)^2), \quad (\text{A.8})$$

and as in standard evaluation of Feynman diagrams, we can reduce tensor discontinuities to scalar one by any reduction procedure.

Mass dependent integrals can be analyzed quite easily in the light cone frame.

For illustrative purposes consider

$$\begin{aligned} & J_{34}(s_1, s_2, s, t, m_1, m_2) \\ &= \int d^4 k \frac{\delta_+(k^2) \delta_+((p_2 - k)^2 - m_2^2) \delta_+((p_1 - k)^2 - m_1^2)}{(p_1 - k + q_1)^2}. \end{aligned} \quad (\text{A.9})$$

In this integral, the  $k^\pm$  components of the internal momentum are fixed at the values

$$k^+ = \frac{q(-2m_1Q^2 + 2m_2Q^2 + m_1s_2 - 2Q^2s_2 + s_1s_2)}{-4Q^4 + s_1s_2}, \quad (\text{A.10})$$

$$k^- = \frac{q(2m_1Q^2 - m_1s_1 + m_2s_1 + 2Q^2s_1 - s_1s_2)}{4Q^4 - s_1s_2}. \quad (\text{A.11})$$

We get

$$J_{34}(s_1, s_2, s, t) = (-2\pi i)^3 \frac{Q^2}{2(4Q^4 - s_1s_2)} T_{ang} \quad (\text{A.12})$$

with

$$\begin{aligned} T_{ang} &= \int_0^{2\pi} \frac{d\theta}{A + B\cos[\theta]} \\ &= 2 \frac{\pi}{(A_2^2 - B_2^2)^{1/2}} \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} A_2 &= s + m_1 - 2k^+p_1^- - 2k^-p_1^+ - 2k^+q_1^- - 2k^-q_1^+; \\ B_2 &= \left(8q_1^+q_1^-(2k^+k^- - m_1)\right)^{1/2}. \end{aligned} \quad (\text{A.14})$$

Remarkably  $T_{ang}$  is a rational function of  $Q^2(s_1, s_2, t)$  in the Breit frame even for nonvanishing  $m_1$  and  $m_2$ . This is due to the identity

$$A^2 - B^2 = \frac{w(m_i, Q^2, s_1, s_2)}{(-4Q^4 + s_1s_2)^2} \quad (\text{A.15})$$

$$\begin{aligned} w(m_i, Q^2, s_1, s_2) &= \left( -4m_1Q^4 + 2m_2Q^2s + 4Q^4s + 2m_1Q^2s_1 \right. \\ &\quad \left. - 2m_2Q^2s_1 - 2Q^2ss_1 - 2m_1Q^2s_2 - 2Q^2ss_2 \right. \\ &\quad \left. - m_1s_1s_2 + ss_1s_2 \right)^2 \end{aligned} \quad (\text{A.16})$$

which gives

$$J_{34}(s_1, s_2, s, t) = \frac{\pi}{2\delta W(m_1, m_2)}, \quad (\text{A.17})$$

and is the result given in eq. (38).

## Appendix B. Radiative Corrections

The evaluation of eq. (53) proceeds as follows. The integrations in the  $l^-$ ,  $l_\perp^2$ , and in the angular variable  $\phi$  in the transversal plane can be done explicitly. We define

$$\cos \phi = \frac{|q_\perp \cdot l_\perp|}{|q_\perp||l_\perp|}, \quad (\text{B.1})$$

and since  $q_{1\perp} = q_{2\perp}$ , we obtain the relation  $q_{12\perp} \cdot l = 0$ . The only angular dependence in  $\phi$ , in the integral, is contained in the factor  $(q_1 - l)^2$ . Therefore we obtain

$$\begin{aligned} I_{5b} &= -\frac{1}{4q_{12}^+} \int_{a0}^{a1} dl^+ \int_0^{2\pi} d\phi \frac{\Delta_2(l^+, l_m)}{((q_1^+ l_m + q_1^- l^+ - 2|q_\perp||l_\perp|\cos\phi)} \\ &= \frac{\pi}{4q_{12}^+} \int_{a0}^{a1} dl^+ \frac{\Delta_2(l^+, l_m)}{((q_1^+ l^- + q_1^- l^+)^2 - 4q_1^+ q_1^- l^+ l_m)^{1/2}}, \end{aligned}$$

which gives, after integration, eq. (55).

$\mu_1$  and  $\mu_2$  are momenta at the boundary, fixed by the conditions on the energy flow. In order to fix these two momenta let's rewrite these conditions in the light cone variables

$$\begin{aligned} l^+ + l_m &> 0, \\ q_1^+ + l^+ + q_1^- + l^- &> 0, \\ q_{12}^+ + l^+ + q_{12}^- + l^- &> 0, \\ s' = (p_1 - l)^2 &= s_1 - 2p_1^+ l^- - 2p_1^- l^+ > 0. \end{aligned} \quad (\text{B.2})$$

In the last condition above, which is simply the requirement of  $s' > 0$ , we have for simplicity neglected the contributions coming from the regulator mass  $\mu$  and from  $m$  (specifically  $(s' > (m + \mu)^2)$ ). This approximation is not going to affect our conclusions in any significant way.

The analysis of these conditions is quite involved. Here we summarize the basic results.

In the Breit frame these conditions are polynomial constraints in the variable  $Q^2$  and can be re-stated respectively in the form

$$\begin{aligned} l^+ &> \mu_1, \\ l^+ &> \mu_2, \\ l^+ &> \mu_3, \\ l^+ &< \mu_4, \end{aligned} \quad (\text{B.3})$$

where

$$\mu_1 = \frac{(-2Q^2 + s_1)(-2Q^2 + s_2)}{2Q(4Q^4 - s_1 s_2)(-4Q^4 + s_1 s_2)^2(16Q^8 - 16Q^6 s + 8Q^4 s^2 - 4Q^2 s s_1 s_2 + s_1^2 s_2^2)}, \quad (\text{B.4})$$

$$\mu_2 = \frac{Q(s - 2Q^2)(-2Q^2 + s_2)(-16Q^6 + 8Q^4s + 4Q^4s_1 - 4Q^2ss_1 + 4Q^4s_2 + s_1^2s_2 - s_1s_2^2)}{(-4Q^2 + s_1 + s_2)(-4Q^4 + s_1s_2)^2}, \quad (\text{B.5})$$

$$\mu_3 = \frac{(-2Q^2 + s_2)P_3(Q, s_1, s_2, s, t)}{2Q(4Q^2 - s_1 - s_2)(-4Q^4 + s_1s_2)^2}, \quad (\text{B.6})$$

and where

$$\begin{aligned} P_3(Q, s_1, s_2, s, t) = & (-32Q^{10} + 32Q^8s - 16Q^6s^2 - 16Q^6ss_1 + 8Q^4s^2s_1 \\ & + 16Q^8s_2 + 8Q^4ss_1s_2 + 8Q^4s_1^2s_2 - 4Q^2ss_1^2s_2 \\ & - 8Q^4s_1s_2^2 - 2Q^2s_1^2s_2^2 + s_1^2s_2^3), \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \mu_4 = & 2Q^3(-2Q^2 + s_2)(16Q^8 - 16Q^6s + 8Q^4s^2 + 8Q^4ss_1 - \\ & 4Q^2s^2s_1 - 4Q^2ss_1s_2 - 4Q^2s_1^2s_2 + 2ss_1^2s_2 + s_1^2s_2^2). \end{aligned} \quad (\text{B.8})$$

It is possible to prove the validity of the following inequalities

$$0 < \mu_2 < \mu_3 < \mu_1 < l^+ < \mu_4 \quad (\text{B.9})$$

and hence the integration over the variable  $l^+$  in (B.2) is cut from below by  $\mu_1$  and from above by  $\mu_4$ .

## Appendix C. Sudakov methods in $n$ spacetime dimensions

In this appendix we cover some of the technical derivation of the  $n$  dimensional discontinuities, obtained by using a special technique which is essentially based on the use of  $n - 2$  transversal dimensions for the Sudakov variables in the Breit frame [19].

The evaluation of the longitudinal parts of the integrals proceeds in the same way as illustrated in Appendix B. Therefore the  $k^\pm$  components of the momenta inside each loop of integration are fixed. The evaluation of these longitudinal parts brings in a jacobian, which is polynomial in the momentum  $Q$ , which is evaluated on the two dimensional subspace of  $n^\pm$ . The transversal integral, instead, is expressed in terms of  $n - 2$  angular variables. Of these,  $n - 3$  can be integrated trivially, while the remaining one, which we call  $\theta$ , is non trivial.

The angular integral so generated are finite, but we need there expression up to  $O(\epsilon)$ , since also these terms contribute to the finite part of the spectral densities. We illustrate here the derivation of  $\sigma_1$  and  $\sigma_2$ , as defined by eqs. (76) and (77).

We obtain

$$\sigma_1 = \frac{1}{A^2} J(Q, s_1, s_2) \omega(n) T_{ang,1}(n, v) \quad (C.1)$$

where

$$J(Q, s_1, s_2) = \frac{Q^2}{2(4Q^4 - s_1 s_2)} \quad (C.2)$$

is the two-dimensional jacobian, while

$$\omega(n) = 2 \frac{\pi^{n/2-3/2}}{\Gamma[n/2 - 3/2]} \quad (C.3)$$

is a volume factor obtained from integration over  $n - 3$  angular dimensions. The nontrivial part is contained in

$$T_{ang,1}(n, v) = \int_0^{2\pi} d\theta \frac{\sin \theta^{n-4}}{(1 + v \cos \theta)^2}, \quad (C.4)$$

which is of hypergeometric form

$$T_{ang,1}(n, v) = \frac{2^{n-5}}{(1+v)^2} \frac{\Gamma[n/2 - 3/2]^2}{\Gamma[n-3]} F(2, n/2 - 3/2, n-3, \frac{2v}{1+v}). \quad (C.5)$$

Using the quadratic (Gauss) relation

$$F(a, b, 2b, z) = (1 - \frac{z}{2})^{-a} F(1/2a, a/2 + 1/2, b + 1/2, \left(\frac{z}{2-z}\right)^2) \quad (C.6)$$

we can re-express (C.5) in the form

$$T_{ang}(n, v) = 2^{n-5} \frac{\Gamma[n/2 - 3/2]^2}{\Gamma[n-3]} F(1, 3/2, n/2 - 1, v). \quad (C.7)$$

Notice that, being  $v \ll 1$ , the hypergeometric function in (C.7) can be approximated, for all the practical purposes, by the first 2 terms, namely up to order  $v$ .

$\sigma_2$  can not be reduced to a complete explicit expression, because of the logarithmic corrections, which are not exactly of the hypergeometric form. This contribution can be directly evaluated with  $n$  set to be 4 from the beginning, since these corrections already appear with a factor of  $\epsilon$  away from  $n = 4$ .

For this second integral, after that the longitudinal integration has been performed, we obtain

$$\sigma_2 = \frac{J(Q^2, s_1, s_2)}{A^2} \omega(n) T_{ang,2}(v, n), \quad (C.8)$$

where

$$\begin{aligned}
T_{ang,2} &\equiv \int_0^\pi d\theta \sin \theta^{n-4} \frac{\text{Log}[(A + B \cos \theta)/\mu^2]}{(1 + v \cos \theta)^2} \\
&= \text{Log}\left[\frac{A}{\mu^2}\right] T_{ang,1}(n, v) + T_{ang,3}(n, v).
\end{aligned} \tag{C.9}$$

We have defined

$$T_{ang,3} \equiv \frac{2^{n-5}}{(1+v)^2} \int_0^1 dt \text{Log}[(1+v)(1-zt)] t^{n/2-3/2} (1-t)^{n/2-5/2} (1-zt)^{-2}, \tag{C.10}$$

and  $z = 2v/(1+v) \approx 2v$ . Expanding the logarithmic part to lowest order as  $\text{Log}[1-zt] \approx -zt + O(v^2)$  and taking  $n = 4$ , we get

$$T_{ang,3} \approx \text{Log}[1+v] T_{ang,1}(4, v) + \frac{2v}{(1+v)} B[3/2, 1/2] + O(v^2) \tag{C.11}$$

Combining all the terms together we finally obtain

$$T_{ang,2} = \text{Log}\left[\frac{A(1+v)}{\mu^2}\right] T_{ang,1}(4, v) + \frac{\pi v}{(1+v)} + O(v^2). \tag{C.12}$$

Using these results for the angular integrals, it is straightforward to obtain the explicit expressions of  $\Delta^{5d}$  and  $\Delta^{6a}$ .

## Appendix D. Borel methods

In this appendix we present a discussion of how and of when the Borel method [1] can be applied to Compton scattering. We have organized the material starting from 2 point functions. Then we come to the pion form factor [1]. A simple way to invert Borel transformed amplitudes for 3-point functions is presented. Then we show that the method can be extended to lowest order Compton scattering. In this case it is shown that the divergence of the Schwinger representation in the box diagram can be kept under control by working in the euclidean region.

To make our discussion self-contained we first illustrate briefly the simplest application of this technique [1] to correlators of lower rank.

Let's consider the dispersion relation of the polarization operator

$$\Pi(q_1^2) = \frac{1}{\pi} \int_0^\infty \frac{\Delta(s) ds}{(s - q_1^2)} + \text{subtractions}, \tag{D.1}$$

with a singularity cut starting at  $q_1^2 = 0$ . Eq. (D.1) can also be written in the form

$$\Pi(q_1^2) = \frac{1}{\pi} \int_0^\infty ds \int_0^\infty d\alpha \Delta(s) e^{-\alpha(s-q_1^2)}, \quad q_1^2 < 0, \quad (\text{D.2})$$

where we have used the exponential parametrization of the denominator.

The Borel transform in one variable is defined in its differential version by the operator [12]

$$B(Q^2 \rightarrow M^2) = \lim_{\substack{Q^2, n \rightarrow \infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!} (Q^2)^n \left(-\frac{d}{dQ^2}\right)^n. \quad (\text{D.3})$$

$M^2$  denotes the Borel mass.

It satisfies the identity

$$B(Q^2 \rightarrow M^2) e^{-\alpha Q^2} = \delta(1 - \alpha M^2) \quad \alpha Q^2 > 0. \quad (\text{D.4})$$

Acting on the polarization operator  $\Pi(q_1^2)$  with the Borel transform we get the usual exponential suppression of the higher states

$$M^2 B(-q_1^2 \rightarrow M^2) \Pi(q_1^2) = \frac{1}{\pi} \int_0^\infty ds \Delta(s) e^{-s/M^2}. \quad (\text{D.5})$$

At this point we can Borel transform once again eq. (D.5), with respect to the inverse Borel mass  $1/M^2$ , in order to obtain

$$B(1/M^2 \rightarrow \nu) (M^2 B(-q_1^2 \rightarrow M^2) \Pi(q_1^2)) = \frac{1}{\nu} \Delta(1/\nu). \quad (\text{D.6})$$

Therefore, by acting iteratively with Borel transforms on  $\Pi(q_1^2)$ , we obtain an expression from which we can easily identify the spectral weight  $\Delta(s)$  of eq. (D.1).

Let's now turn our attention to vertex functions (see also Fig. 1c).

The Borel transformed amplitude for the pion form factor (the complete spectral density) is given by [1]

$$\begin{aligned} \phi(M_1^2, M_2^2, q^2) &= \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \rho_{\pi 3}^{pert}(s_1, s_2, q^2) e^{-s_1/M_1^2 - s_2/M_2^2} \\ &= \frac{3}{2\pi^2(M_1^2 + M_2^2)} \int_0^1 dx x(1-x) \exp\left(\frac{-xq^2}{(1-x)(M_1^2 + M_2^2)}\right). \end{aligned} \quad (\text{D.7})$$

To isolate the spectral function from eq. (D.7) we need to use Borel transforms and act on it with the differential operator

$$B(1/M_1^2 \rightarrow 1/\nu_1) B(1/M_2^2 \rightarrow 1/\nu_2) M_1^2 M_2^2. \quad (\text{D.8})$$



Here we discuss a possible way of doing this. The inversion of (D.7) can be obtained by taking inverse Laplace transforms twice - respect to  $1/M_i^2$  - of this equation. In fact, for a given function  $F(M^2)$ , the following identity

$$L^{-1}(1/M^2 \rightarrow \nu)F(M^2) = (1/\nu)B(1/M^2 \rightarrow 1/\nu)F(M^2) \quad (D.9)$$

relates the differential operator given in (D.3) to the inverse Laplace transform

$$L^{-1}(1/M^2 \rightarrow \nu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d(1/M^2) \exp(\nu/M^2). \quad (D.10)$$

Defining  $1/M_1^2 = \mu_1, 1/M_2^2 = \mu_2$ , then, we are required to act with the operator defined in eq. (D.8) on the integral function

$$\chi(\mu_1, \mu_2, q^2) = \frac{1}{\mu_1 + \mu_2} \int_0^\infty dx \frac{x}{(x+1)^4} \exp\left(-xQ^2\mu_2 + \frac{xq^2\mu_2^2}{\mu_1 + \mu_2}\right). \quad (D.11)$$

Using the gaussian relation

$$\exp(\alpha^2/(4k)) = (k/\pi)^{1/2} \int_{-\infty}^\infty d\sigma \exp(-k\sigma^2 - \alpha\sigma) \quad (D.12)$$

we can rewrite eq. (D.11) into the form

$$\begin{aligned} \chi(\mu_1, \mu_2, q^2) &= \int_{-\infty}^\infty d\sigma \int_0^\infty dx \frac{x \exp(-xq^2\sigma - (\mu_1 + \mu_2)\sigma^2 - 2x^{1/2}q\mu_2\sigma)}{(x+1)^4(\pi(\mu_1 + \mu_2))^{1/2}}. \end{aligned} \quad (D.13)$$

By using the relation

$$L^{-1}(\mu_1 \rightarrow \nu_1) \frac{1}{(\mu_1 + \mu_2)^{1/2}} \exp(-\mu_1 + \mu_2\sigma^2) = \left(\frac{\pi}{(\nu_1 - \sigma^2)}\right)^{1/2} \Theta(\nu_1 - \sigma^2) \quad (D.14)$$

on (D.13) we finally get

$$\begin{aligned} &L^{-1}(\mu_2 \rightarrow \nu_2)L^{-1}(\mu_1 \rightarrow \nu_1)\chi(\mu_1, \mu_2, q^2) \\ &= L^{-1}(\mu_2, \nu_2) \int_0^\infty \frac{xdx}{(x+1)^4} \int_{-\sqrt{\nu_1}}^{\sqrt{\nu_1}} d\sigma \frac{\exp((-xq^2\mu_2 - 2x^{1/2}q\mu_2\sigma - \mu_2)\nu_1)}{(\nu_1 - \sigma^2)^{1/2}} \\ &= \int_{-\sqrt{\nu_1}}^{\sqrt{\nu_1}} d\sigma \frac{(-\sigma + f^{1/2})^3 q^4}{(\sigma - f^{1/2})^2 + q^2} f^{1/2}(\nu_1 - \sigma^2)^{1/2} \\ &= (1/6)q^4(d/dq^2)^3 Y(q^2, \nu_1, \nu_2), \end{aligned} \quad (D.15)$$

where

$$Y(q^2, \nu_1, \nu_2) = \int_{-\sqrt{\nu_1}}^{\sqrt{\nu_1}} d\sigma \frac{(\sigma - f^{1/2})^3}{f^{1/2}(\nu_1 - \sigma^2)^{1/2} ((\sigma - f^{1/2})^2 + q^2)},$$

$$f = \sigma^2 - \nu_1 - \nu_2. \quad (\text{D.16})$$

By redefining  $\nu_i = s_i$ ,  $i = 1, 2$ , and after doing the explicit integration of in eq. (D.16), it is easy to relate this last result to the 3-particle cut integral for the triangle diagram (Fig. 1b), in the scalar case,

$$\frac{Y(q^2, s_1, s_2)}{4q^2} = \Delta_3(s_1, s_2, t) \quad (\text{D.17})$$

The derivative with respect to  $q_1^2$  in eq. (D.15) takes into account the fermionic character of the propagators in Fig. 1b compared to the scalar case (see eq. (D.17)). The spectral function for the pion form factor can then be expressed in the form [1]

$$\begin{aligned} \rho_{\pi 3}^{\text{pert}}(s_1, s_2, t) \\ = \frac{3}{2\pi^2} t^2 \left( \left( \frac{d}{dt} \right)^2 + \frac{t}{3} \left( \frac{d}{dt} \right)^3 \right) \frac{1}{((s_1 + s_2 - t)^2 - 4s_1 s_2)^{1/2}} \end{aligned} \quad (\text{D.18})$$

with  $t = -q^2$ .

For four-point functions this procedure simplifies considerably. The method can be applied exactly as in the form factor case, although it is necessary to work in the euclidean region from the beginning. However 1) the amplitude has *first* to be continued in the unphysical  $s$  and  $t$  region before any Borel transform is taken 2) afterwards the result has to be continued back to the physical region. These two steps allow us to bypass problems related to the divergences of the Schwinger representation (see the discussion below) of the amplitude. Since, by doing so, we are comparing *timelike* and *spacelike* representations of the same amplitude, we analyze step by step the effect of the Borel transforms *both* on the timelike amplitude and on the spacelike one. As we have already emphasized, the method works *only* if the dispersion relation is unbound, i.e. the dispersion integral can be extended to infinity for both  $s_1$  and  $s_2$ . We have pointed out that for massless correlators this is indeed the case.

Let's work in the euclidean region of  $T_4(p_1^2, p_2^2, s, t)$  with spacelike external invariants ( $p_1^2, p_2^2 < 0$ ,  $s = (p_1 + q_1)^2 < 0$ ,  $t = (q_2 - q_1)^2 < 0$ ). It is then possible to relate  $T_4$  to its euclidean continuation  $T_{4E}$  and use the Schwinger parametrization for the latter:

$$T_{4E} = \int d^4k \int_0^\infty [d\alpha_i] \exp \left( -\alpha k^2 - \beta(p'_1 - k)^2 - \gamma(p'_1 - k + q'_1)^2 - \epsilon(p'_2 - k)^2 \right), \quad (\text{D.19})$$

where  $\alpha_i$  is a short notation for the proper time parameters. Divergences, in this representation, reappear when we move into the physical  $s, t$  region. We perform the integration over the loop momentum in eq. (D.19) to get

$$T_{4E} = \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) \int_0^\infty \Sigma^3 d\Sigma e^\tau \quad (\text{D.20})$$

where

$$\tau = -\Sigma(A_1(x_i)s_1 - A_2(x_i)s_2 - A_3(x_i, t)) \quad (D.21)$$

with suitable expression for  $A_1, A_2, A_3$ . In the region where  $p_1^2 = -p_1'^2$ ,  $p_2^2 = -p_2'^2$  and, in general,  $p_i \cdot p_j = -p_i' \cdot p_j'$ ,  $T_{4E}$  is related to  $T_4$  by an analytic continuation

$$T_4(p_1^2, p_2^2, s, t) = iT_{4E}(p_1'^2, p_2'^2, s', t') \quad (D.22)$$

In the euclidean region the form of the dispersion relation becomes

$$T_{4E} = -\frac{i}{4\pi^2} \int_0^\infty ds_1' \int_0^\infty ds_2' \frac{\Delta(-s_1', -s_2', s', t')}{(s_1' + p_1'^2)(s_2' + p_2'^2)} + \text{subtr.} \quad (D.23)$$

Exactly as in the case of the polarization operator (eqs. (D.1) and (D.2)), we can now apply Borel transforms on eq. (D.23) to get

$$\begin{aligned} \eta_E(M_1^2, M_2^2, s', t') &= B(p_1'^2 \rightarrow 1/M_1^2) B(p_2'^2 \rightarrow 1/M_2^2) T_{4E}(p_1'^2, p_2'^2, s', t') \\ &= -\frac{i}{4\pi^2} \int_0^\infty ds_1' \int_0^\infty ds_2' e^{-s_1'/M_1^2 - s_2'/M_2^2} \Delta(-s_1', -s_2', s, t) \\ &= \pi^2 \int_0^1 dx_1 dx_2 dx_3 \delta(1 - A_1 \Sigma M_1^2) \delta(1 - A_2 M_2^2 \Sigma) e^{-\Sigma A_3 \Sigma^3} d\Sigma \\ &= \pi^2 \int_0^1 \frac{dx_1 dx_2 \pi^2}{M_1^2 M_2^2 x_1 x_2} e^{-b_1/M_1^2 - b_2/M_2^2} \end{aligned} \quad (D.24)$$

$$(D.25)$$

where we have defined

$$\begin{aligned} b_1 &= s \frac{(1 - x_1 - x_2)}{x_2} \\ b_2 &= \frac{(-sx_1 + tx_2)}{x_1}. \end{aligned} \quad (D.26)$$

Notice that  $B(p_i'^2 \rightarrow M_i^2) = B(-p_i^2 \rightarrow M_i^2)$ , since  $p_i^2 < 0$  by assumption.

By acting similarly on  $T_4$  as given by (22) - with  $B(-p_i^2 \rightarrow 1/M_i^2)$  - we also get

$$\begin{aligned} \eta(M_1^2, M_2^2, s, t) &= -\frac{1}{4\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \Delta(s_1, s_2, s, t) e^{-s_1/M_1^2 - s_2/M_2^2} \\ &= B(-p_1^2 \rightarrow M_1^2) B(-p_2^2 \rightarrow M_2^2) T_4(p_1^2, p_2^2, s, t). \end{aligned} \quad (D.27)$$

Reapplying the Borel transform (this time with respect to  $1/M_i^2$ ) sequentially on both  $\eta$  and  $\eta_E$  and using eq. (D.4) we get respectively

$$B(1/M_1^2 \rightarrow \nu_1) B(1/M_2^2 \rightarrow \nu_2) (M_1^2 M_2^2 \eta_E(M_1^2, M_2^2, s', t'))$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \frac{dx_1 dx_2}{x_1^2 x_2} \delta(1 - b_1 \nu_1) \delta(1 - b_2 \nu_2) \\
&= -\frac{1}{4\pi^2 \nu_1 \nu_2} \Delta_E(-1/\nu_1, -1/\nu_2, s', t')
\end{aligned} \tag{D.28}$$

for the euclidean part, and

$$\begin{aligned}
&B(1/M_1^2 \rightarrow \nu_1) B(1/M_2^2 \rightarrow \nu_2) (M_1^2 M_2^2 \eta(M_1^2, M_2^2, s, t)) \\
&= -\frac{1}{4\pi^2 \nu_1 \nu_2} \Delta(1/\nu_1, 1/\nu_2, s, t).
\end{aligned} \tag{D.29}$$

in the physical region. As we can see from eq. (D.28), the integral on the two parameters  $x_1$  and  $x_2$  is now trivial and can be performed straightforwardly. A simple calculation then gives

$$\Delta_E(-1/\nu_1, -1/\nu_2, s' t') = \frac{4\pi^4}{\nu_1 \nu_2 s' t'}. \tag{D.30}$$

Using the identification  $1/\nu_1 \rightarrow s_1, 1/\nu_2 \rightarrow s_2$ , analytically continuing back (the euclidean result in (D.30) with the prescriptions  $s' \rightarrow -s, t' \rightarrow -t$  and using eq. (D.22) we finally get

$$\Delta(p_1^2, p_2^2, s, t) = -4i \frac{\pi^4}{st}, \tag{D.31}$$

valid in the physical region. This result agrees with the one obtained from the cutting rules eq. (21). The Borel method allows to overcome, in this simpler case, the difficulties related to the evaluation of discontinuity integrals. The original discussion of this section therefore provides an extension of the method to Compton scattering, from the known case of the form factor [1]. As we have extensively discussed, however, the presence of  $u$ -channel singularities makes it inapplicable to Compton scattering in full generality.

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## Figure Captions

**Fig. 1.** (a) The Compton scattering amplitude; (b) the 3-particle cut for Compton scattering. One internal line is off shell. (c) The triangle singularity for the form factor to lowest order.

**Fig. 2.** The integration contours for the scalar amplitude in the  $p_1^2$  plane, which includes the  $u$ -channel threshold (thick black line) .

**Fig. 3.** The  $u$ -channel cut of the box diagram.

**Fig. 4.** (a) The diagram for the gluonic power corrections. (b) The 3-particle cut for the same diagram; (c) its  $u$  channel cut.

**Fig. 5.** (a) Radiative correction to the box diagram in the scalar amplitude. The line with the encircled "1" denoting a scalar of mass  $\mu$ . (b) A cut generated by (a).

(c) Its decomposition as a convolution of a 2-particle cut in the  $s'$  energy (bottom part) with a 2-to-2 on-shell scattering amplitude. (d) The cut giving the imaginary contribution generated by (a) and (e) its factorized form.

**Fig. 6** (a) A subdiagram of vertex type with a 2-particle cut. (b) Its factorized form. (c) The self-energy 2-particle cut. (c) The imaginary contribution to the spectral density obtained by the insertion of (b).

**Fig. 7** The complete OPE of the imaginary parts of the double spectral densities for Compton scattering.

**Fig. 8** (a) A diagram containing a cut of type  $t$ . (b) The discontinuity of  $t$  type extracted from (a). (c) A possible (real) cut which is not factorizable. (d) The same as in (b). Notice that 3 propagators are off-shell.

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